CONVERGENCE IN DISTRIBUTION OF PRODUCTS OF RANDOM MATRICES: A SEMIGROUP APPROACH

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ABSTRACT. The problem of weak convergence of the sequence of convolution powers of a probability measure has been considered in this paper in the general context of a noncompact semigroup and in particular, in the semigroup of nonnegative and real matrices. Semigroup methods have been used to give simple proofs of some recent results of Kesten and Spitzer in nonnegative matrices. It has been also shown that these methods often lead to similar results in the more general context of real matrices.

1. **Introduction.** Random matrices have recently attracted the attention of many probabilists. Recent results in [**KS**] are essentially concerned with the problem of weak convergence of the convolution iterates of a probability measure on a locally compact semigroup of $d \times d$ nonnegative matrices (under multiplication and usual topology) to a probability measure different from the unit mass at the zero matrix. We show here how some of the main results in [**KS**] can be obtained using certain semigroup ideas. Our methods here result in more general results for matrices with arbitrary real entries. This is the main motivation for this paper, since, as Kesten and Spitzer mention in [**KS**, p. 364, line 3], their methods do not apply to the case of real matrices.

2. **Preliminaries.** Completely simple semigroups play an important role in this paper. A semigroup is completely simple if it is simple and has a primitive idempotent. A locally compact completely simple topological semigroup K is topologically isomorphic (see [**MT**]) to the product structure $X \times G \times Y$, where $X = E(Ke)$, $G = eKe$ and $Y = E(eK)$. Here $e = e^2$ is a fixed idempotent element of K and $E(A)$ is the set of all idempotents in A. X, G and Y are given relative topologies from K and the multiplication in $X \times G \times Y$ is defined by

$$(x, g, y)(x', g', y') = (x, gyx'g', y').$$

The set $X \times G \times Y$ is given the product topology. It is well known that a compact semigroup has a closed kernel (that is, a minimum 2-sided ideal), which is completely simple. Mukherjea and Tserpes (see [**MT**]) proved that the support of an idempotent probability measure on a locally compact (second countable Hausdorff) semigroup is a closed completely simple subsemigroup with a compact group factor. It is also known [**C**] that if a semigroup S (of $d \times d$ matrices over a field) has a completely simple kernel K, then $K = m(S)$, where $m(S)$ is the set of all those

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matrices in S with the minimal rank. Another result that will play an important role here is Theorem 4.3 in [**M**, p. 189]. In this theorem, the author gave a necessary and sufficient condition for the weak convergence of a tight sequence (P^n) of the iterates of a probability measure on a noncompact semigroup S. It is therefore relevant to understand when (P^n) is tight in a general noncompact semigroup. In this context, let us state the following (rather important) theorem without proof.

THEOREM 2.1. *Let S be a second countable locally compact Hausdorff topological semigroup and P be a probability measure on the Borel sets of S such that $S = \mathrm{cl}(\bigcup_{n=1} S_P^n)$. Then the convolution sequence (P^n) is tight iff S has a completely simple kernel K with a compact group factor such that for every open set G containing K, $\lim_{n\to\infty} P^n(G) = 1$.*

3. The convergence problem in nonnegative matrices. Here J will denote the set of all $d \times d$ nonnegative matrices which have at least one zero row or one zero column or both. J^c is the complement of J. Our first theorem is the simplest and perhaps the most important result in this paper. Though this result does not appear in [**KS**], it is influenced by the results in [**KS**].

THEOREM 3.1. *Let S be the closed semigroup generated by the support S_P of a probability measure on $d \times d$ nonnegative matrices. Then $P_n \equiv (1/n)\sum_{k=1}^{\infty} P^k$ converges weakly to Q in $P(S)$ and $S_Q \cap J^c$ is nonempty iff S is compact and $m(S) \cap J^c$ is nonempty.*

PROOF. Suppose that P_n converges weakly to Q. Then $Q = Q^2$ and S_Q is the completely simple kernel of S. Let $x \in S_Q \cap J^c$. Then xS_Qx is a compact group and $xS_Qx \subset J^c$. Let $e = e^2 \in xS_Qx$. Then $eSe = eS_Qe$ is a compact group of nonnegative matrices and therefore, finite. Since $e \in J^c$, e has a strictly positive diagonal (see [**M**$_1$]). Since eSe is finite, there is a positive number s such that for any $y \in S$ and for any i,j $(1 \le i,j \le d)$, $e_{ii}y_{ij}e_{jj} \le s$ so that $y_{ij} \le s/(e_{ii}e_{jj})$. This means that S is compact. Since $m(S)$ is an ideal, $S_Q \subset m(S)$. This proves the "only if" part. The "if" part is immediate from Theorem 4.3 in [**M**]. □

Let us remark that if S contains the zero matrix, then the sequence (P^n) converges weakly to $\delta_{\{0\}}$ iff (P^n) is tight. This is immediate from Theorem 4.3 in [**M**]. The next theorem covers a substantial part of Theorem 1 and Theorem 2 in [**KS**]. We give a different proof using semigroups.

THEOREM 3.2. *Let P be a (regular) probability measure on $d \times d$ nonnegative matrices such that the following two conditions hold:*
(KS_1) $P(J) = 0$;
(KS_2) $P^m(I) > 0$ *for some positive integer m, where I is the set of all strictly positive $d \times d$ matrices.*
Let S be the closed semigroup (under multiplication and usual topology) generated by S_P. Then the following statements are equivalent:
(A) $P^n \to Q \in P(S)$ *weakly and $S_Q \cap J^c \ne \varnothing$.*
(B) *For each $u > 0$, there exists $k_u > 0$ such that for each positive integer n, $P^n(\mathcal{B}_u) > 1 - u$, where $\mathcal{B}_u = \{B \in S: \min_{i,j} B_{i,j} \le k_u\}$ and there exists $v > 0$ and a subsequence (n_t) such that for $t \ge 1$, $P^{n_t}(\mathcal{A}_v) > v$, where $\mathcal{A}_v = \{B \in S: \max_{i,j} B_{ij} > v\}$.*
(C) (P^n) *is tight and $P^n \nrightarrow \delta_{\{0\}}$ weakly.*

(D) *Let* $M_n = A_1 A_2 \cdots A_n$, *where* (A_i) *is an iid sequence in* S *with common distribution* P. *Then there exists* $w > 0$ *such that*

$$\Pr(\{(M_n)_{ij} \leq w \text{ for all } i, j, \text{ all } n \geq 1\}) = 1$$

and also, the second condition in (B) *holds.*

PROOF. (A)$\Rightarrow$(B) Trivial.

(B)$\Rightarrow$(C) Assume (B). Write $I_k = \{B \in S \colon B_{ij} \geq 1/k \text{ for all } i, j\}$. Since I is an ideal of J^c, $P^n(I)$ is nondecreasing with n by (KS_1) and by a simple argument given on p. 38 on [**MT**], using (KS_2),

$$(3.1) \qquad \lim_{n \to \infty} P^n(I) = 1.$$

Thus there exists a positive integer N and $k_0 \geq 1$ such that

$$(3.2) \qquad P^N(I_{k_0}) > 1/2.$$

Let $0 < u' < 1$ and $u = u'/4$. Then for any positive integer s,

$$(1 - u')/4 \leq \Pr(M_{2N+s} \in \mathcal{B}_u, M_N \in I_{k_0}, A_{N+s+1} \cdots A_{2N+s} \in I_{k_0})$$
$$\leq \tfrac{1}{4} \cdot P^s(W_{k_0, u'}),$$

where $W_{k_0, u'} = \{B \colon \max_{i,j} B_{ij} \leq k_0^2 k_u\}$. This establishes (C).

(C)$\Rightarrow$(D) Immediate from Theorem 3.1 and Theorem 4.3 [**M**].

(D)$\Rightarrow$(C) Trivial.

(C)$\Rightarrow$(A) Assume (C). By Theorem 4.3 [**M**], $(1/n)\sum_{k=1}^{n} P^k \to Q \in P(S)$ and S_Q is the completely simple kernel of S with a compact group factor. Clearly $0 \notin S$ by (C). Since $I \supset I \cdot S_Q \cdot I$ and for each positive integer n, $P^n * Q * P^n = Q$, $Q(I) \geq P^n(I)Q(S_Q)P^n(I)$ for each n. By (3.1), $Q(I) = 1$. Let $x \in S_Q \cap I$. Then $xS_Q x \subset I$. But $xS_Q x$ is a compact group and therefore, it must be a singleton. By Theorem 4.3 [**M**], (A) follows.

We remark that under conditions of Theorem 3.2, we can also show easily that the following are equivalent: (i) (P^n) is tight and $0 \notin S$. (ii) P^n converges weakly to $Q \in P(S)$ and $Q(I \cap S) = 1$. (iii) S is compact and $0 \notin S$. (iv) $m(S)$ is bounded and $m(S) \neq \{0\}$. Note that boundedness of $m(S)$ implies the boundedness of its closure. Then $\mathrm{cl}(m(S))$ is compact and has a kernel, which must equal $m(\mathrm{cl}(m(S)))$ by [**C**]. Hence, $m(S)$ is closed. If $m(S) \neq \{0\}, 0 \notin S$. Then, $I \cap m(S) \supset (I \cap S)m(S)(I \cap S)$. As in Theorem 3.1, it follows that S is compact.

In the rest of this section, we give a semigroup proof and extension of Theorem 3 [**KS**, p. 367]. This theorem is the most interesting (and perhaps, the most difficult) theorem in [**KS**]. It is stated in [**KS**] without a proof. It is completely covered by our Theorem 3.2 ((A)$\Leftrightarrow$(B)) and our next theorem.

THEOREM 3.3. *Under assumptions and notations of Theorem* 3.2, *the following two conditions are equivalent*:

(1) *There exists* $\varepsilon > 0$, $p > 0$ *and a subsequence* (n_k) *such that for some* i, $1 \leq i \leq d$,

$$(3.3) \qquad \liminf_{k \to \infty} \Pr\left\{p \leq \sum_{j=1}^{d}(M_{n_k})_{ij} \leq \frac{1}{p}\right\} \geq \varepsilon.$$

(2) $P^n \to Q$ *weakly and* $Q(I) = 1$.

It is relevant and useful to give first a semigroup version of Theorem 3.3, which will make this theorem crystal clear when $S \subset J^c$.

THEOREM 3.4. *Let P be a probability measure on a locally compact second countable Hausdorff topological semigroup S such that $S = \mathrm{cl}(\bigcup_{n=1}^{\infty} S_P^n)$. Suppose the following condition holds:*

For compact subsets A and B, the sets

(3.4)
$$AB^{-1} = \{y \colon yx \in A \text{ for some } x \in B\},$$
$$A^{-1}B = \{y \colon xy \in B \text{ for some } x \in A\}$$

are compact. Suppose also that there is a compact subset $K_0 \subset S$, a subsequence (n_k) and $\varepsilon > 0$ such that

(3.5)
$$P^{n_k}(K_0) > \varepsilon \quad \text{for each } k \geq 1.$$

Then the sequence (P^n) is tight.

PROOF. Let $H = \{x \in S \colon \sum_{n=1}^{\infty} P^n(N_x) = \infty$ for every open set N_x containing $x\}$. By (3.5), H is nonempty. Also, H is an ideal of S, that is, $HS \subset H$ and $SH \subset H$. By Theorem 5.9 [**MT**, p. 191], H is a closed completely simple kernel of S. [Note that the conditions (CR) and (CL) mentioned in the theorem there are implied by (3.4).] Now we show that

(3.6)
$$\sup_{K \text{ compact}} \limsup_{n \to \infty} \sup_{x \in S} P^n(Kx^{-1}) = 1.$$

The idea of the proof is not new. See Theorem 4.14, p. 51 in [**MT**]. For completeness, we prove (3.6). Let b be the sup in (3.6). Then $b > 0$ by (3.5). Suppose, if possible, that $b < 1$. Choose $c = b + [b(1 - b)/(1 + 2b)]$. Then $b < c < 1$ and $c(1 + c) < 2b$. Let B be any compact set. There exist a positive integer N, a compact set A and $0 < q < c/2$ such that

(3.7)
$$P^N(S - A) < c/2 - q$$

and

(3.8)
$$\sup_{z \in S} P^N(Bz^{-1}) < c - q.$$

Let $E = A^{-1}B$. Then E is compact by (3.4). For $n > N$ and any $x \in S$,

$$P^n(Bx^{-1}) = \int P^N(Bx^{-1}y^{-1})P^{n-N}(dy) = \int_{Ex^{-1}} + \int_{S-Ex^{-1}}$$
$$\leq (c - q)P^{n-N}(Ex^{-1}) + (c/2 - q)[1 - P^{n-N}(Ex^{-1})]$$
$$\leq (c + c^2)/2 < b.$$

This is a contradiction. This proves (3.6).

By (3.6), there exist a compact set K, elements a_n and a positive integer n_0 such that for $n > n_0$,

(3.9)
$$P^n(Ka_n^{-1}) > 1 - \varepsilon/2.$$

By (3.4) and (3.9), $n_k > n_0 \Rightarrow Ka_{n_k}^{-1} \cap K_0 \neq \varnothing$. This means that $a_{n_k} \in K_0^{-1}K$. Thus, if $B = K(K_0^{-1}K)^{-1}$, then B is compact by (3.4) and for $n_k > n_0$,

(3.10)
$$P^{n_k}(B) > 1 - \varepsilon/2.$$

Thus, (P^{n_k}) is tight. Now by employing exactly the same method as given in Step III of the proof of Theorem 4.14 ([**MT**, p. 50]), it follows that there is a subsequence $(m_k) \subset (n_k)$ such that

(3.11) $$P^{m_{k+1}-m_k} \to Q = Q^2 \in P(S) \quad \text{weakly.}$$

For $x \in S_Q$ and any open N_x containing x,

$$0 < Q(N_x) \le \liminf_{k \to \infty} P^{m_{k+1}-m_k}(N_x).$$

Thus, $S_Q \subset H$. Note also that $P * Q = Q * P$ so that $\overline{S_P S_Q} = \overline{S_Q S_P}$. This means that $\overline{SS_Q} = \overline{S_Q S} \subset H$, since H is an ideal and $S_Q \subset H$. Since $\mathrm{cl}(SS_Q)$ is also an ideal of S and H is simple, $H = \mathrm{cl}(SS_Q)$. Thus, the closed completely simple semigroup H is generated by the support of $P * Q$. Moreover,

(3.12) $$(P * Q)^{m_{k+1}-m_k} = P^{m_{k+1}-m_k} * Q \to Q * Q = Q,$$

so that $(P * Q)^n$ does not converge to zero vaguely. By Theorem 4.14 ([**MT**, p. 50]), the group factor of H is compact. Also, it follows easily that given $\varepsilon > 0$, there is a compact set $D \subset H$ and a positive integer m_0 such that for $n \ge m_0$, $P^n * Q(D) = (P * Q)^n(D) > 1 - \varepsilon$. Thus, if C is compact so that $Q(C) > 1 - \varepsilon$, then for $n \ge m_0$, $P^n(DC^{-1}) > 1 - 2\varepsilon$. Thus, (P^n) is tight. $\square$

Let us remark that when S is a closed semigroup of nonnegative matrices, then (3.4) holds for $A \subset J^c$ and $B \subset J^c$. Thus, when $S \subset J^c$, Theorem 3.3 is immediate from Theorem 3.4. When $S \cap J \ne \varnothing$, there will be complications in the proof of Theorem 3.3, but the main ideas are those we came across in Theorem 3.4. In what follows, we prove Theorem 3.3 completely. Let P and S be as in Theorem 3.3. Instead of assuming (3.3), we assume the weaker condition

(3.13) $$P^{n_k}(B(q_1, q_2)) \ge \varepsilon, \quad \text{for each } k \ge 1,$$

where $q_1 > 0$, $q_2 > 0$ and $\varepsilon > 0$, and

$$B(q_1, q_2) = \left\{ A \colon \max_{i,j} A_{ij} \ge q_2 \text{ and for each } s, \ \min_i A_{is} \le q_1 \right\}.$$

Write $I^u = \{A \colon 1/u \le A_{ij} \le u \text{ for every } i, j\}$. Then using (3.1), we see that there exist a positive integer N and $v > 0$ such that

(3.14) $$P^N(I^v) > 1 - \varepsilon/4.$$

Using (3.13) and (3.14), we have for $n_k > N$,

$$\varepsilon \le P^{n_k}(B(q_1, q_2)) = \int P^{n_k - N}(x^{-1}B(q_1, q_2))P^N(dx)$$

$$\le P^{n_k - N}((I^v)^{-1}B(q_1, q_2)) + \varepsilon/4.$$

Now it can be easily verified that

$$(I^v)^{-1}B(q_1, q_2) \subset F(q_1, q_2, v),$$

$$F(q_1, q_2, v) = \left\{ A \in S \colon q_2/vd \le \max_{i,j} A_{ij} \le q_1 v \right\}.$$

Thus, it follows that for $n_k > N$,

(3.15) $$P^{n_k - N}(F(q_1, q_2, v)) \ge 3\varepsilon/4.$$

Now one can also verify that there are $a > 0$ and $b > 0$ such that

$$I^v \cdot F(q_1, q_2, v) \cdot I^v \subset \{A \in S : a \le A_{ij} \le b \text{ for all } i, j\}.$$

Let us call the set on the right W. Then it follows from (3.14) and (3.15) that for $n_k > N$,

$$(3.16) \qquad P^{n_k + N}(W) \ge (1 - \varepsilon/4)^2 \cdot (3\varepsilon/4).$$

Now we need the following lemma.

LEMMA 3.1. *Suppose that $W \subset I$, W is compact and $P^{n_k}(W) > q > 0$, for each $k \ge 1$. Then (P^{n_k}) is a tight sequence. Also, given $\varepsilon > 0$, there is a positive integer n_0 and a compact set $W_\varepsilon \subset I$ such that $\lim_k \sup P^{n_k + n_0}(W_\varepsilon) > 1 - \varepsilon$.*

PROOF. Let $1 \le i \le d$. Write

$$\beta_u = \left\{ A \in S : u \le \sum_{j=1}^d A_{ij} \le \frac{1}{u} \right\}.$$

We first show that

$$(3.17) \qquad \sup_{u > 0} \lim_{n \to \infty} \sup_{x \in S} P^n(x^{-1}\beta_u) = 1.$$

Notice that the set W is contained in some β_u. It can also be verified easily that for each $u > 0$, the set $\beta_u (I^v)^{-1}$ is contained in $\beta_{u'}$, for some $u' > 0$. The proof of (3.17) can now follow exactly the same lines as in the proof of (3.6). Here we only use

$$P^n(x^{-1}\beta_u) = \int P^N(y^{-1}x^{-1}\beta_u) P^{n-N}(dy)$$

$$= \int_{x^{-1}\beta_{u'}} + \int_{S - x^{-1}\beta_{u'}}.$$

The details are the same as before. So we may and do take (3.17) for granted. By (3.17), given $0 < \varepsilon < q$, there exist $u > 0$ and elements x_m in S such that for all sufficiently large m and k,

$$P^m(x_m^{-1}\beta_u) > 1 - \varepsilon/2 \quad \text{and} \quad P^{n_k}(W) > q.$$

It follows that for sufficiently large k, $x_{n_k}^{-1}\beta_u \cap W \ne \varnothing$ so that $x_{n_k} \in \beta_u W^{-1} = T_u$, say.

It is now routine to verify the following fact. There exist $a_\varepsilon > 0$ and $b_\varepsilon > 0$ such that

$$(3.18) \qquad T_u^{-1}\beta_u \subset L_\varepsilon \subset B_\varepsilon,$$

where

$$L_\varepsilon = \left\{ A \in S : \max_{i,j} A_{ij} \ge a_\varepsilon \text{ and } \min_{i,j} A_{ij} \le b_\varepsilon \right\}$$

and

$$B_\varepsilon = \left\{ A \in S : \min_{i,j} A_{ij} \le b_\varepsilon \right\}.$$

Thus for each $\varepsilon > 0$, there exists k_0 such that for $k \ge k_0$,

$$(3.19) \qquad P^{n_k}(B_\varepsilon) \ge P^{n_k}(L_\varepsilon) > 1 - \varepsilon.$$

Using (3.14) and the same proof used for (B)⇒(C) in Theorem 3.2, we see that given $0 < u' < 1$, taking $u = u'(1 - \varepsilon/4)^2$,

$$(1 - \varepsilon/4)^2 (1 - u')$$
$$\leq \Pr(M_{2N+(n_k-2N)} \in \mathcal{B}_u, M_N \in I^v \text{ and } A_{N+n_k+1} \cdots A_{2N+n_k} \in I^v)$$
$$\leq (1 - \varepsilon/4)^2 \cdot P^{n_k - 2N}(M_u),$$

where

$$M_u = \left\{ D \in S : \max_{i,j} D_{ij} \leq b_u v^2 \right\}.$$

It follows that (P^{n_k}) is a tight sequence.

Now we note that closure$(I^v \cdot L_\varepsilon \cdot I^v) \subset I$ so that it follows from (3.14), (3.19) and the tightness of (P^{n_k}) that there is a compact set $W_\varepsilon \subset I$ such that for sufficiently large k,

$$(3.20) \qquad P^{n_k + 2N}(W_\varepsilon) > (1 - \varepsilon)^3.$$

Now we continue with the hypothesis of Lemma 3.1. By (3.19), there is a subsequence $m_k \subset (n_k)$ such that $P^{m_k} \to P'$ weakly, where P' is a probability measure different from δ_0. Following the same proof as given in Step III (of Theorem 7.14, p. 54 in [MT]), we see that there is a subsequence $(p_k) \subset (m_k)$ such that for $j \geq 1$,

$$(3.21) \qquad P^{p_k - j} \to P_j \in P(S), \qquad P_0 = P'.$$

(Note that $P_j \in P(S)$, since $(P^{p_k - j})$ is tight. Because of the tightness of (P^{p_k}) and the regularity of P^j, given $\varepsilon > 0$, there are compact sets D_ε and E_ε with $E_\varepsilon \subset J^c$ such that

$$1 - \varepsilon < P^{p_k}(D_\varepsilon) = \int P^{p_k - j}(D_\varepsilon x^{-1}) P^j(dx)$$

$$= \int_{E_\varepsilon} + \int_{S - E_\varepsilon} \leq P^{p_k - j}(D_\varepsilon E_\varepsilon^{-1}) + \varepsilon.$$

Since $E_\varepsilon \subset J^c$, $D_\varepsilon E_\varepsilon^{-1}$ is compact.) Thus, for each $j \geq 1$, $P_j * P^j = P^j * P_j = P'$. Now we show that (P_{p_k}) is tight. Notice that

$$P_{p_k} * P^{p_k + 2N} = P^{p_k + 2N} * P_{p_k} = P' * P^{2N}.$$

We use (3.20). Given $\varepsilon > 0$, there is a compact set $W_\varepsilon \subset I$ such that $P' * P^{2N}(W_\varepsilon) > 1 - 3\varepsilon$ and (3.20) holds so that

$$(3.22) \qquad \begin{aligned} 1 - 3\varepsilon < P' * P^{2N}(W_\varepsilon) &= \int P_{p_k}(x^{-1} W_\varepsilon) P^{p_k + 2N}(dx) \\ &= \int_{W_\varepsilon} + \int_{S - W_\varepsilon} \leq P_{p_k}(W_\varepsilon^{-1} W_\varepsilon) + 3\varepsilon. \end{aligned}$$

Since $W_\varepsilon \subset I$, $W_\varepsilon^{-1} W_\varepsilon$ is compact and (P_{p_k}) is tight. If Q is a weak*-cluster point of (P_{p_k}), then $Q \in P(S)$ and it follows as in the proof of Step III of Theorem 7.14 [MT] that $Q = Q^2$ and there is a subsequence $(q_k) \subset (p_k)$ such that

$$(3.23) \qquad P^{q_{k+1} - q_k} \to Q = Q^2 \in P(S).$$

Notice that by (3.22) and by a similar argument,

$$Q(W_\varepsilon^{-1} W_\varepsilon \cap W_\varepsilon W_\varepsilon^{-1}) > 1 - 12\varepsilon.$$

Since $W_\varepsilon \subset I$, $W_\varepsilon^{-1} W_\varepsilon \cap W_\varepsilon W_\varepsilon^{-1} \subset J^c$. It follows that

(3.24) $$Q(J^c) = 1.$$

Write $s_k = q_{k+1} - q_k$ and $E_v = \{A \in S \colon \max_{i,j} A_{ij} \geq v\}$. By (3.23) and (3.24), there is a positive v such that for sufficiently large k, $P^{s_k}(E_v) > 1 - \varepsilon$. As in (3.14), there are positive integers N, r, v' such that $P^N(I^r) > 1 - \varepsilon$ and $P^{2N}(I^{v'}) > 1 - \varepsilon$. Then

$$1 - \varepsilon < P^{s_k}(E_v) \leq P^{s_k - 2N}((I^{v'})^{-1} E_v) + \varepsilon.$$

Since $(I^{v'})^{-1} E_v \subset E_w$ for some $w > 0$, for sufficiently large k, then

$$P^{s_k - 2N}(E_w) > 1 - 2\varepsilon.$$

Since $\mathcal{D} \equiv \mathrm{closure}(I^r \cdot E_w \cdot I^r) \subset I$, for large k

$$P^{s_k}(\mathcal{D}) > (1 - \varepsilon)^2 (1 - 2\varepsilon).$$

This means that

(3.25) $$Q(I) = 1.$$

Now let $x, y \in S \cap J^c$ be such that $x \notin Sy$. Then it is not difficult to show that there exist open subsets $N_x \subset J^c$, $N_y \subset J^c$ such that $x \in N_x$, $y \in N_y$ and $S \cap (N_x N_y^{-1}) = \varnothing$. By (3.16), there exists $x \in I \cap S$ such that for every open set N_x containing x,

(3.26) $$P^n(N_x) \nrightarrow 0 \quad \text{as } n \to \infty.$$

Following the proof of Theorem 5.9 [**MT**] almost word for word, we have

(3.27) $$y \in Sx \cap J^c \Rightarrow Sx = Sy$$

and

(3.28) $$y \in xS \cap J^c \Rightarrow xS = yS.$$

Notice that if we write $S_1 = S - \{0\}$, then since $x \in I$, we have

$$
\begin{aligned}
xSx &= xS_1 x \cup \{0\} && \text{if } 0 \in S; \\
&= xS_1 x && \text{if } 0 \notin S.
\end{aligned}
$$

In any case, $xS_1 x \subset I$ and $xS_1 x$ is a subsemigroup. Let $y \in xS_1 x$. Using (3.28),

(3.29) $$y \cdot xSx = xSx \quad \text{and therefore} \quad yxS_1 x = xS_1 x.$$

Similarly, using (3.27),

(3.30) $$xS_1 xy = xS_1 x \quad \text{for } y \in xS_1 x.$$

By (3.29) and (3.30), $xS_1 x$ is a group contained in I.

Let us now define the set K_0 by

$$K_0 = \{y \in I \cap S \colon P^n(N_y) \nrightarrow 0 \text{ as } n \to \infty \text{ for every open } N_y \text{ containing } y\}.$$

Then K_0 is a nonempty ideal of $S \cap J^c$. Also, by what we have proven above, for each $y \in K_0$, $y^3 \in yS_1 y$, which is a group. Thus, by [**C**], $m(K_0)$ is a completely simple minimal ideal of K_0. Since every idempotent in I has rank one and since by

(3.28), $y^3 \in yS \Rightarrow y \in y^3 S \subset yS$ and since every element in a group has the same rank, it follows that $m(K_0) = K_0$.

Since $Q(I) = 1$ and for every $y \in S_Q$ and any open N_y containing y,

$$0 < Q(N_y) \leq \liminf_k P^{s_k}(N_y),$$

it is clear that

(3.31) $$Q(K_0) = 1.$$

Since $P * Q = Q * P$ and $Q = Q^2$,

$$(P * Q)^n (K_0) = (P^n * Q)(K_0) \geq (P^n * Q)((S \cap J^c)K_0)$$
$$\geq P^n (S \cap J^c) Q(K_0) = 1 \quad \text{for every } n \geq 1.$$

Restricting Q and $P* Q$ to K_0 (with relative topology) and noticing that $\mathrm{cl}(SS_Q)$ is an ideal of S (since $P * Q = Q * P$) so that $K_0 \subset \mathrm{cl}(SS_Q)$. This means that the support of $P * Q$ restricted to K_0 generates K_0. It follows from Theorem 4.14 in [MT] that the group factor of K_0 is compact in K_0. Since $K_0 \subset I$, the group factor of K_0 is a singleton. This means that $(P * Q)^n = P^n * Q$ converges weakly to some P_0, a probability measure. Thus, given $\varepsilon > 0$, there is a compact set L and a compact E, both contained in I such that for sufficiently large n,

$$1 - \varepsilon < P^n * Q(L) < \int_E P^n (Lz^{-1}) Q(dz) + \varepsilon.$$

Thus, for large n, $P^n(LE^{-1}) > 1 - 2\varepsilon$. Thus, (P^n) is tight and we have proven Theorem 3.3. □

For nonnegative matrices, we can also prove the following theorem (along the lines of [KS], but more general than what is given in [KS]).

THEOREM 3.5. *Let P and S be as in Theorem 3.2. Suppose that $U \in P(S)$, $P * U = U$ and $U(\{B \in S: B \text{ has a zero column}\}) = 0$. Then $P^n \to Q$ weakly, $Q(I \cap S) = 1$, $U * U = U$, $Q * U = U$, $U * Q = Q$ and $S_U \subset S_Q$.*

4. Weak convergence in 2×2 real matrices.

Let P be a regular probability measure on 2×2 real matrices and S be the closed semigroup with respect to multiplication generated by S_P. Suppose $P^n \to Q$ weakly. Then S_Q is the completely simple kernel of S and $Q = Q * P = P * Q$. Also, $S_Q = m(S)$ and the group factor of S_Q is compact. There are now two possibilities:

(I) The rank of the matrices in $m(S)$ is two. Then S_Q has only one idempotent element and it is a compact group. Since $S \cdot S_Q \subset S_Q$, $S = S_Q$. Thus, in this case, the closed semigroup generated by S_P is a compact group and Q is the normed Haar measure on this group.

(II) $S_Q \subset$ rank one matrices. In this case we also assume that $m(S)$ has no subgroup with exactly two elements, so any compact subgroup of $m(S)$ is a singleton. [Note that such a condition was not needed in §3, since a compact group of positive matrices is a singleton.] Under these assumptions, for any idempotent e in S_Q, eS_Qe is a compact subgroup of S_Q and therefore, a singleton. Since $S_Q = S_QeS_Q$, every x in S_Q is of the form yez so that

$$x^2 = y(ezye)z = yez = x.$$

Thus, every element in S_Q has rank one and is idempotent.

Now let us make the following assumption:

(4.1) $$S_Q \cap J^c \neq \varnothing.$$

(Recall that J^c is the set of all 2×2 matrices which do not have any zero rows or columns.) Let us choose e from $S_Q \cap J^c$ so that e can be taken as

(4.2) $$e = \begin{pmatrix} a_1 e_1 & a_1 e_2 \\ a_2 e_1 & a_2 e_2 \end{pmatrix}$$

where

(4.3) a_1, a_2, e_1, e_2 are all nonzero   and $a_1 e_1 + a_2 e_2 = 1$.

(The last equality holds because $e = e^2$.) Recall that

(4.4) $$eSe = eS_Q e = e.$$

We now show that

(4.5)   either $eS = eS_Q = e$ or $Se = S_Q e = e$.

(In other words, for any e in $S_Q \cap J^c$, either every row of e is a common left eigenvector for every matrix in S with eigenvalue 1 or every column of e is a common right eigenvector for every matrix in S with eigenvalue 1.) Notice that $e = e^2 \in S_Q$ which is an ideal of S so that $eS_Q \subset eS = e(eS) \subset eS_Q$. Similarly, $S_Q e = Se$. To establish (4.5), let $y \in eS_Q$ and $x \in S_Q e$ such that

(4.6) $$x = \begin{pmatrix} b_1 x_1 & b_1 x_2 \\ b_2 x_1 & b_2 x_2 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} c_1 y_1 & c_1 y_2 \\ c_2 y_1 & c_2 y_2 \end{pmatrix},$$

where

(4.7) $b_1 x_1 + b_2 x_2 = 1$ and $c_1 y_1 + c_2 y_2 = 1$.

Since $yx = e$ by (4.4) and $e \in J^c$, y cannot have a zero row and x cannot have a zero column. Therefore,

(4.8) $c_1 \neq 0, \quad c_2 \neq 0, \quad x_1 \neq 0, \quad x_2 \neq 0.$

Now we distinguish between three possible cases.

Case 1. Suppose that there is an $x \in S_Q e$ for which $b_1 = 0$. Since $yx = e$, simple matrix multiplication yields that $c_1 y_2 = a_1 e_2$ and $c_2 y_2 = a_2 e_2$. Using (4.3) and (4.7), it follows easily that $y = e$. Thus, in this case, $eS = e$.

Case 2. Suppose that for each x in $S_Q e$, b_1 is nonzero, but there exists $y \in S_Q$ such that $y_2 = 0$. Then for this y in eS_Q and any x in $S_Q e$, $yx = e$ implies (after a little computation) that $x = e$. Thus, in this case, $Se = e$.

Case 3. Suppose that for each $x \in S_Q e$, b_1 is nonzero and for each y in eS_Q, $y_2 \neq 0$. Then, for our convenience, we rewrite y, x and e (y arbitrarily chosen from eS_Q and x from xS_Q) as

$$y = \begin{pmatrix} a & b \\ \frac{a(1-a)}{b} & 1-a \end{pmatrix}, \quad x = \begin{pmatrix} c & d \\ \frac{c(1-c)}{d} & 1-c \end{pmatrix}, \quad e = \begin{pmatrix} k' & k \\ \frac{k'(1-k')}{k} & 1-k' \end{pmatrix},$$

where $b \neq 0$, $d \neq 0$, $k \neq 0$. Now simple computations using $yx = e$ give

(4.9) $$\begin{array}{c} ad + b(1-c) = k, \quad (c/d)[ad + b(1-c)] = k; \\ \{(1-a)/b\}[ad + b(1-c)] = 1 - k'. \end{array}$$

Simplifying the equation in (4.9), we have

(4.10) $$(k - d)(k - b) = 0.$$

If $k = b$, then we have from (4.9) that

$$c/d = a/b = k'/k,$$

which means that $y = e$. This means that if there is an x in $S_Q e$ such that $x \neq e$, then $eS_Q = eS = e$. Similarly, if $k = d$, (4.5) must again hold.

Thus, we have established (4.5) in all possible cases under the assumption of (4.1). Now we consider the case when (4.1) no longer holds. We still assume, of course, the condition of (II). We will show that even in this case, (4.5) holds.

Recall that by (II), every element in S_Q is still idempotent. First, let

(4.11) $$e \in S_Q \quad \text{and} \quad e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$Se = S_Q e \subset \left\{ \begin{pmatrix} 0 & r \\ 0 & 1 \end{pmatrix} : r \text{ a real number} \right\}$$

and

$$eS = eS_Q \subset \left\{ \begin{pmatrix} 0 & 0 \\ r & 1 \end{pmatrix} : r \text{ a real number} \right\}.$$

Using the fact that $eS_Q e = e$, it follows after computations that either

$$eS = eS_Q = e \quad \text{and} \quad S \subset \left\{ \begin{pmatrix} s & t \\ 0 & 1 \end{pmatrix} : s, t \text{ reals} \right\}$$

or

$$Se = S_Q e = e \quad \text{and} \quad S \subset \left\{ \begin{pmatrix} s & 0 \\ t & 1 \end{pmatrix} : s, t \text{ reals} \right\}.$$

Next, let $e \in S_Q$ be of the form $\begin{pmatrix} 0 & r \\ 0 & 1 \end{pmatrix}$, $r \neq 0$.

It can be shown again that in this case either

$$Se = S_Q e = e \quad \text{or} \quad eS = eS_Q = e$$

and

$$S \subset \left\{ \begin{pmatrix} s & t \\ 0 & 1 \end{pmatrix} : s, t \text{ reals} \right\}.$$

There is no point into going into every other possible case. Let us just mention that it can be verified in exactly the same manner that (4.5) holds in all the other cases. Thus, we have proven the following Kesten-Spitzer type theorem for real 2×2 matrices.

THEOREM 4.1. *Let P be a probability measure on 2×2 real matrices and S be the closed semigroup generated by S_P. Then the following results hold:*
   (a) *If $P^n \to Q \in P(S)$ weakly, then*
      (i) $S_Q = m(S)$;
      (ii) $Q = \delta_0$ *iff $0 \in S$;*
      (iii) *the minimal rank of the matrices in S is two iff S is a compact group of invertible matrices and Q is the normed Haar measure on S;*

(iv) *the minimal rank of the matrices in S is one and $m(S)$ contains no subgroup with exactly two elements iff there is a nonzero idempotent rank one matrix x in S such that $xS = x$ or $Sx = x$.*

(b) *Suppose that $\lim_{n\to\infty} P^n(G) = 1$ for every open set $G \supset m(S)$. If the rank of the matrices in $m(S)$ is two, then*

(i) *$P^n \to 0$ vaguely when S is noncompact;*

(ii) *when S is compact, S is a compact group and then P^n converges weakly to the normed Haar measure on S iff there is no proper normal subgroup H of S such that $S_P \subset gH$ for some $g \in S - H$. If the rank of the matrices in $m(S)$ is one, then $P^n \to Q$ weakly and $S_Q = m(S)$ when there is a nonzero vector x such that $xS_P = x$ or $S_P x^T = x^T$.*

5. The convergence problem in real matrices. Here we derive analogs of earlier results in $d \times d$ matrices with arbitrary real entries for arbitrary finite d. First, we need a lemma.

LEMMA 5.1. *Suppose that $x = x^2$ is a $d \times d$ real matrix with no zero row. Suppose that there are two rank one $d \times d$ real matrices y and z such that*

$$(5.1) \qquad\qquad xy = x \quad and \quad xz = x.$$

Then, $y = y^2, z = z^2$ and $yz = y$.

PROOF. Notice that x has rank one. Write x as $x = (x_{ij}) = (a_i x_j)$, where each a_i is nonzero (since x has no zero row) and not all the x_j's are zero. Write y as $y = (y_{ij}) = (b_i y_j)$. Since $xy = x$, we have for $1 \le i, j \le d$,

$$(5.2) \qquad\qquad a_i x_j = \sum_{k=1}^{d} a_i x_k b_k y_j.$$

Since $x = x^2$ and $x \ne 0$,

$$(5.3) \qquad\qquad \sum_{i=1}^{d} a_i x_i = 1.$$

It follows from (5.2) that

$$(5.4) \qquad\qquad \sum_{i=1}^{d} a_i x_i = \left(\sum_{i=1}^{d} a_i y_i \right) \cdot \left(\sum_{k=1}^{d} b_k x_k \right).$$

Cancelling a_i and then multiplying by b_j in (5.2), we have

$$b_j x_j = b_j y_j \cdot \sum_{k=1}^{d} b_k x_k$$

so that

$$(5.5) \qquad\qquad \sum_{j=1}^{d} b_j x_j = \left(\sum_{j=1}^{d} b_j y_j \right) \left(\sum_{k=1}^{d} b_k x_k \right).$$

It follows from (5.3), (5.4) and (5.5) that

(5.6) $$\sum_{k=1}^{d} b_k x_k \neq 0 \quad \text{and} \quad \sum_{j=1}^{d} b_j y_j = 1.$$

This means that $y = y^2$.

Similarly, we write z as $z = (z_{ij}) = (c_i z_j)$. Since $xz = x$, it follows as before that $z = z^2$. It follows from (5.2) that

(5.7) $$y_j = 0 \Rightarrow x_j = 0;$$

also, $x_j/y_j = \sum_{k=1}^{d} b_k x_k$.

Let us now prove that $yz = y$. First, notice that

(5.8) $$(yz)_{ij} = b_i z_j \cdot \sum_{k=1}^{d} c_k y_k.$$

As in (5.7), we also have

$$x_j = 0, \quad \text{whenever } z_j = 0;$$

and

(5.9) $$x_j/z_j = \sum_{k=1}^{d} c_k x_k, \quad \text{whenever } z_j \neq 0.$$

Suppose that $y_j = 0$. Then $x_j = 0$ by (5.7). Since $xz = x$,

$$a_i x_j = a_i z_j \cdot \sum_{k=1}^{d} b_k x_k$$

and therefore, since by (5.6), $\sum_{k=1}^{d} b_k x_k \neq 0$, we have $z_j = 0$. Then $(yz)_{ij} = 0$ by (5.8). Thus, when $y_j = 0$, $(yz)_{ij} = y_{ij} (= b_i y_j)$.

Now suppose that $y_j \neq 0$. By (5.2) and (5.6),

(5.10) $$x_j = y_j \cdot \sum_{k=1}^{d} b_k x_k \neq 0.$$

Therefore, by (5.9), $z_j \neq 0$ and

(5.11) $$0 \neq x_j = z_j \cdot \sum_{k=1}^{d} c_k x_k.$$

It follows also from (5.2) that

(5.12) $$\sum_{s=1}^{d} c_s x_s = \left(\sum_{s=1}^{d} c_s y_s \right) \cdot \left(\sum_{k=1}^{d} b_k x_k \right).$$

By (5.10)–(5.12),

(5.13) $$y_j = z_j \cdot \sum_{s=1}^{d} c_s y_s.$$

By (5.8) and (5.13),
$$(yz)_{ij} = b_i y_j = y_{ij}.$$
This proves that $yz = y$. $\quad\square$

In [**KS**], it was shown for 2×2 nonnegative matrices that under (KS_1) and (KS_2) (see Theorem 4.1), whenever $P^n \to Q$ weakly and $Q \neq \delta_0$, then there exists $x = (x_1, x_2)$, $x_1 > 0$, $x_2 > 0$, such that either $xS_P = x$ or $S_P x^T = x^T$. Here we show that this result is no longer valid for higher dimensional matrices.

EXAMPLE 5.1. Consider the semigroup S of 3×3 strictly positive matrices given by $S = \{e_0, e_1, e_2, e_3\}$, where

$$e_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix}, \qquad e_1 = \begin{pmatrix} \frac{3}{4} & \frac{1}{12} & \frac{1}{6} \\ \frac{3}{4} & \frac{1}{12} & \frac{1}{6} \\ \frac{3}{4} & \frac{1}{12} & \frac{1}{6} \end{pmatrix},$$

$$e_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 1 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{1}{12} & \frac{1}{6} \end{pmatrix}, \qquad e_3 = \begin{pmatrix} \frac{3}{4} & \frac{1}{12} & \frac{1}{6} \\ \frac{3}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{3}{8} & \frac{1}{24} & \frac{1}{12} \end{pmatrix}.$$

Then the multiplication table for S is given by

$\times$	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_0	e_1
e_1	e_0	e_1	e_0	e_1
e_2	e_2	e_3	e_2	e_3
e_3	e_2	e_3	e_2	e_3

The semigroup S is completely simple since it is finite and has no proper ideals. Also, every element of S has rank one. Consider the probability measure P on S given by
$$P(e_i) = 1/4, \qquad i = 0, 1, 2, 3.$$
(Actually, any P such that $S_P = S$ will do.) Since every subgroup of S is a singleton, by Theorem 2.1, $P^n \to Q$ and $S_Q = S$.

Now suppose that there exists a nonzero vector $u = (u_1, u_2, u_3)$, at least one u_i being nonzero, such that $uS = u$. Choose $a \neq 0$, $b \neq 0$ and $c \neq 0$ such that $cu_1 + au_2 + bu_3 = 1$. Define the matrix x by

$$x = \begin{pmatrix} cu_1 & cu_2 & cu_3 \\ au_1 & au_2 & au_3 \\ bu_1 & bu_2 & bu_3 \end{pmatrix}.$$

Then $x = x^2$ and $xS = x$. It follows then from Lemma 5.1 that for y, z in S, $yz = y$. This contradicts the above multiplication table.

Despite the above example, half of Theorem 3 [**KS**, p. 384] holds even for general $d \times d$ ($d \geq 2$) real matrices as we show in Theorem 5.1. It should be pointed out that for a vector x and a set $\mathcal{A}$ of matrices, the equation $x\mathcal{A} = x$ does no longer imply that $\mathcal{A}$ is bounded unless x is strictly positive and the matrices in $\mathcal{A}$ nonnegative. Also note that for nonnegative matrices, the conditions (KS_1) and (KS_2) are equivalent to the condition that $\lim_{n \to \infty} P^n(I) = 1$. This is because of the way we proved (3.1) and because

$$P^{2m}(J_r) \geq \int_{J_r} P^m(x^{-1} J_r) P^m(dx) = P^m(J_r)$$

since $S \subset x^{-1}J_r$ for each $x \in J_r$, and

$$P^{2m}(J_c) \geq \int_{J_c} P^m(J_c x^{-1})P^m(dx) = P^m(J_c)$$

since $S \subset J_c x^{-1}$ for every x in J_c, where

$$J_r = \{A : A \text{ has at least one zero row}\}$$

and

$$J_c = \{A : A \text{ has at least one zero column}\}.$$

I is also an open ideal of J^c containing all rank one matrices in J^c. Under (KS_1) and (KS_2), for P^n to converge weakly to Q in nonnegative matrices, it is necessary that $S_Q \subset$ rank one matrices and $\lim_{n \to \infty} P^n(G) = 1$ for every open set G containing rank one matrices. Considering all these, it is now clear that condition (5.14) below is for real matrices what conditions (KS_1) and (KS_2) are for nonnegative matrices.

THEOREM 5.1. *Let P be a probability measure on $d \times d$ real matrices such that for every open set $G \supset I$, where in real matrices I denotes the set of rank one matrices,*

(5.14)
$$\lim_{n \to \infty} P^n(G) = 1.$$

Suppose that for some $x = (x_1, x_2, \ldots, x_d)$, not all x_i's zero, $xS_P = x$ or $S_P x^T = x^T$. Then $P^n \to Q$ weakly and $S_Q \subset I$.

PROOF. First we need to prove that (P^n) is tight. Let S be the closed semigroup generated by S_P. Since $xS = x$, S has no zero. If $P^m(I) = 0$ for each positive integer m, then noting that $I \cap S$ is a closed subset of S it follows immediately from Lemma 7.7 in [HM] that (P^n) is tight. Let us then assume that $P^m(I) > 0$ for some positive integer m. Then as in Theorem 4.3, it follows that

(5.15)
$$\lim_{n \to \infty} P^n(I) = 1.$$

Let $xS_P = x$, where x is a nonzero vector. Then $xS = x$. There is a rank one $d \times d$ idempotent matrix x_0 such that x is its first row. Then $x_0 S = x_0$. Since $x_0 \neq 0$, $0 \notin S$. Now notice that

(5.16)
$$y, z \in I \cap S \Rightarrow yz = y$$

by Lemma 5.1. Let $p > 0$. There exists $q > 0$ and a positive integer N such that

(5.17)
$$n \geq N \Rightarrow P^n(I \cap S) > 1 - q > \sqrt{1 - p}.$$

Let K_p be a compact set $\subset I \cap S$ such that $P^n(K_p) > 1 - q$. Let $n \geq N$. Then using (5.16),

$$P^{n+N}(K_p) = \int P^N(K_p y^{-1})P^n(dy)$$

$$\geq \int_{S \cap I} P^N(K_p y^{-1})P^n(dy)$$

$$= P^N(K_p)P^n(S \cap I) > (1 - q)^2 > 1 - p.$$

This proves that the sequence (P^n) is a tight sequence. By Theorem 4.3 of [M], $(1/n) \sum_{k=1}^n P^k \to Q \in P(S)$. Since $S \cap I$ is closed, $S_Q \subset S \cap I$. Since $xS = x$ and x is nonzero, S cannot contain any subgroup of the form $\{-1, 1\}$. It follows that the group factor of S_Q being compact is a singleton. By Theorem 4.3 of [M], $P^n \to Q$ weakly. □

EXAMPLE 5.2. Consider the semigroup $S_0 = K_1 \cup K_2$ of 2×2 nonnegative matrices, where

$$K_1 = \left\{ \begin{pmatrix} 1 & m \\ 0 & n \end{pmatrix} : m, n \text{ are nonnegative integers} \right\}$$

and

$$K_2 = \left\{ \begin{pmatrix} 1 & n \\ 0 & 0 \end{pmatrix} : n \text{ is a nonnegative integer} \right\}.$$

Notice that the set K_2 is a right zero semigroup so that for any $e \in K_2$, $eK_2e = e$. It is clear that K_2 is the completely simple kernel of S_0. Notice that if $x = (1, 0)$, then $s_0 x^T = x^T$. Consider a probability measure P on S_0 such that

$$S_P \supset \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

It is clear that Theorem 5.1 applies and $P^n \to Q$ weakly, where $S_Q \subset K_2$ (actually, $S_Q = K_2$ if S_P contains also $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$). Notice that neither S_Q nor the closed semigroup generated by S_P is compact, unlike in Theorem 3.1. □

Finally we state a general theorem for real matrices.

THEOREM 5.2. *Let P be a probability measure on $d \times d$ real matrices and $S = \text{cl}(\bigcup_{n=1}^\infty S_P^n)$. Suppose that for each open set $G \supset m(S)$, $\lim_{n\to\infty} P^n(G) = 1$. Then the following results hold:*

(a) $P^n \to Q$ weakly and $S_Q = m(S)$ iff $m(S)$ is completely simple with a compact group factor G such that for any proper normal subgroup H of G, $gS_P e \cap (G - H) \neq \varnothing$ for each $g \in G - H$, where e is the identity of G.

(b) If the rank of the matrices in $m(S)$ is one and $m(S)$ does not contain a subgroup with exactly two elements, then P^n converges weakly iff $m(S)$ is completely simple with a compact group factor.

(c) If for each $x \in S$, there is a power x^m lying in a subgroup of S, then either $P^n \to 0$ vaguely or (P^n) is tight.

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